Programming and Reasoning with Infinite Data in Isabelle/HOL

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You can run our material on your computer

Isabelle: `isabelle.in.tum.de` Theories: `is.gd/Isabelle`

Based on joint work with

Jasmin Blanchette

VU VRIJE UNIVERSITEIT AMSTERDAM

Dmitriy Traytel

ETH zürich

... who have also contributed slides for this presentation

Context

Proof assistants = Tools for

- Defining mathematical concepts
- Proving facts about them

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$$\text{collatz} : \text{Nat} \to \text{LazyList}(\text{Nat})$$

$$\text{collatz}(n) = \begin{cases} [] & \text{if } n \leq 1 \\ \text{collatz}(n \,/\, 2) & \text{if } n > 1 \text{ and } n \text{ even} \\ n \mathbin{\#\#} \text{collatz}(3 * n + 1) & \text{if } n > 1 \text{ and } n \text{ odd} \end{cases}$$

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recursion

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Big proofs about programs in Isabelle

(an incomplete list)

seL4

Microkernel

Klein et al.

CAVA

LTL model checker

Lammich, Nipkow et al.

Markov_Models

pCTL model checker

Hölzl, Nipkow

Flyspeck

Programs in Hales's proof of the Kepler conjecture

Bauer, Nipkow, Obua

CoCon

Conference management system

Kanav, Lammich, Popescu

PDF-Compiler

Probability density functions compiler

Eberl, Hölzl, Nipkow

JinjaThreads

Java compiler & JMM

Lochbihler

IsaFoR/CeTA

Termination proof certifier

Sternagel, Thiemann et al.

IsaSAT

SAT solver with WL

Fleury, Blanchette et al.

Codatatypes and Corecursion

Codatatypes/Corecursion in Isabelle

Programming Language Examples

Types with infinite values

type	**finite values**	**infinite values**
nat	0, 1, 2, 3, …	
enat	0, 1, 2, 3, …	∞

Types with infinite values

type	**finite values**	**infinite values**
	$0, S(0), S(S(0)), S(S(S(0))), \ldots$	
nat	0, 1, 2, 3, …	
enat	0, 1, 2, 3, …	∞
	$0, S(0), S(S(0)), S(S(S(0))), \ldots$	$S(S(S(S(S(\ldots)))))$

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	$0, S(0), S(S(0)), S(S(S(0))), \ldots$	$S(S(S(S(S(\ldots)))))$
list	[], [0], [0,0], [0,1,2,3,4], ...	
stream		[0,0,0,...], [1,2,3,...], [0,1,0,1,...], ...
llist	[], [0], [0,0], [0,1,2,3,4], ...	[0,0,0,...], [1,2,3,...], [0,1,0,1,...], ...

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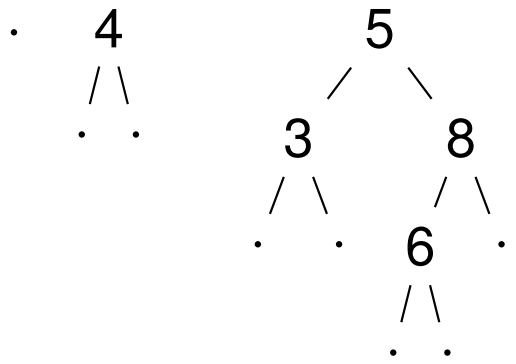
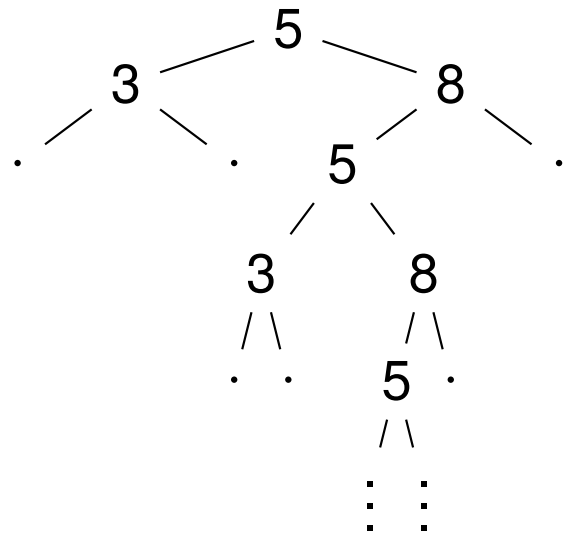
stream		[0,0,0,…], [1,2,3,…], [0,1,0,1,…], …
llist	[], [0], [0,0], [0,1,2,3,4], …	[0,0,0,…], [1,2,3,…], [0,1,0,1,…], …

tree





No recursion on codatatypes

```
datatype   nat  = 0    | Suc  nat
codatatype enat = Zero | eSuc enat
```

Suppose we could do recursion on codatatypes ...

```
  to_nat Zero     = 0
| to_nat (eSuc n) = Suc (to_nat n)
```

No recursion on codatatypes

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Suppose we could do recursion on codatatypes . . .

```
  to_nat Zero     = 0
| to_nat (eSuc n) = Suc (to_nat n)
```

. . . but codatatypes are not well-founded: ∞ = eSuc ∞

```
to_nat ∞ = to_nat (eSuc ∞) = Suc (to_nat ∞)
```

$$n = 1 + n$$

False

Building infinite values

Recursion

- datatype as argument
- peel off one constructor
- recursive call only on arguments of the constructor

Corecursion

- codatatype as result
- produce one constructor
- corecursive call only in arguments to the constructor

```
∞ = eSuc ∞
```

Computing with infinite values

Computing with codatatypes is pattern matching on results

Definition (Productivity)

We can inspect arbitrary finite amounts of output in finitely many steps.

Am I productive?

```
s = 0 ## s
```

```
s = 0 ## s
```

primitive corecusion

```
$ ghci
Prelude> s = 0 : s
Prelude> take 5 s
[0,0,0,0,0]
Prelude>
```

```
s = 0 ## stl s
```

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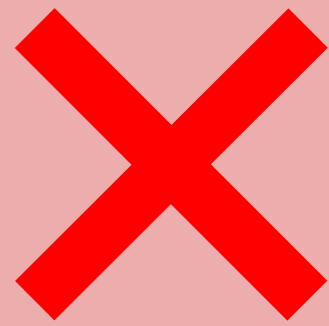

`stl` evil

```
$ ghci
Prelude> s = 0 : tail s
Prelude> take 5 s
[0^CInterrupted
Prelude>
```

```
s = 0 ## 1 ## s
```

```
s = 0 ## 1 ## s
```

corecursion up to constructors

```
eo s = shd s ## eo (stl (stl s))
```

```
eo s = shd s ## eo (stl (stl s))
```

primitive corecusion

```
s = 0 ## 1 ## eo s
```

```
s = 0 ## 1 ## eo s
```

eo evil

$$s \oplus t = (shd\ s + shd\ t) \mathbin{\#\#} (stl\ s \oplus stl\ t)$$

$$s \oplus t = (\mathrm{shd}\ s + \mathrm{shd}\ t) \mathbin{\#\#} (\mathrm{stl}\ s \oplus \mathrm{stl}\ t)$$

primitive corecusion

$$s \otimes t = (shd\ s * shd\ t) \mathbin{\#\#} (stl\ s \otimes t \oplus s \otimes stl\ t)$$

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corecursion up to $\oplus$

$$(\sigma \otimes \tau)(n) = \sum_{k=0}^{n} \binom{n}{k} \times \sigma(n-k) \times \tau(k)$$

The standard definition

$$s = 0 \mathbin{\#\#} ((1 \mathbin{\#\#} s) \oplus s)$$

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corecursion up-to constructors and $\oplus$

$$s = (0 \mathbin{\#\#} 1 \mathbin{\#\#} s) \oplus (0 \mathbin{\#\#} s)$$

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corecursion up to constructors and $\oplus$

$\oplus$ comes before the guard

$$s = (1 \mathbin{\#\#} s) \otimes (1 \mathbin{\#\#} s)$$

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corecursion up to constructors and $\otimes$

$\otimes$ comes before the guard

```
selfie s = shd s ## selfie (selfie (stl s) ⊕ selfie s)
```

```
selfie s = shd s ## selfie (selfie (stl s) ⊕ selfie s)
```

corecursion up to $\oplus$ and `selfie` [sic!]

```
s m n =  if (m == 0 && n > 1) || gcd m n == 1
         then n ## s (m * n) (n + 1)
         else s m (n + 1)
```

```
s m n =  if (m == 0 && n > 1) || gcd m n == 1
         then n ## s (m * n) (n + 1)
         else s m (n + 1)
```

mixed recursion/primitive corecursion

```
s n =  if n > 0
       then stl (s (n - 1)) ⊕ (0 ## s (n + 1))
       else 1 ## s 1
```

```
s n =  if n > 0
       then stl (s (n - 1)) ⊕ (0 ## s (n + 1))
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```

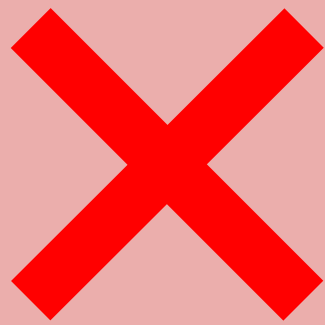

`stl` really evil

Codatatypes and Corecursion

Codatatypes/Corecursion in Isabelle

Programming Language Examples

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Our Line of Work

Foundational framework for

defining all the green stuff and more

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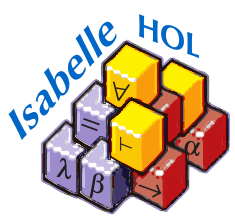
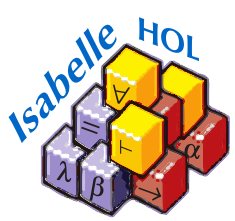
in an LCF-style proof assistant (

)

Our Line of Work

Foundational **framework** for

defining **all** the green stuff **and more**

in an **LCF-style** proof assistant 

Burden on the **user**: prove { parametricity or termination } here and there

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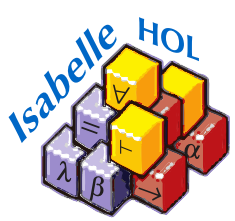
in an **LCF-style** proof assistant (

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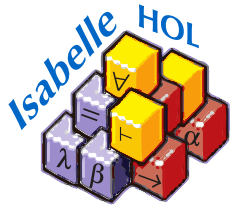
Burden on the **user**: prove { parametricity or termination } here and there

Most of the time: **automatic**

Guiding Principle

LCF Philosophy: Reduce everything to a small trusted kernel

Kernel of 

$\approx$
- simply typed lambda calculus $+$
- classical higher-order logic (axioms) $+$
- nonrecursive constant definition $+$
- nonrecursive type definition

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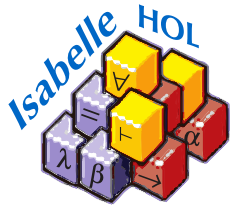
Kernel of 

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Our agenda make Isabelle/HOL a (co)recursion-friendly environment

LICS'12 ITP'14 IJCAR'14 ESOP'15 ICFP'15 ESOP'17 LICS'17

Related Work

Guarded Corecursion

- FRP (Krishnaswami & Benton, …)	type system
- clocks (Atkey & McBride)	type system
- guards (Clouston et al.)	type system
- abstract GSOS (Milius et al.)	category theory
- companions (Pous & Rot)	category theory (“up-to”)

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Coq, Lean	constructor guarded corecursion	built-in
Agda	copatterns + sized types	built-in + type system
	up-to techniques, Thu@POPL 	
Dafny	mixed recursion/corecursion	built-in

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Isabelle’	corecursion up-to *friendly* operations mixed with recursion	smart corecursor + wellfounded recursion

Primitive Corecursor

`codatatype` *Stream* = *Int ## Stream*

Primitive Corecursor

$\texttt{codatatype}\ \mathit{Stream} = \mathit{Int}\ \#\#\ \mathit{Stream}$

– $\mathit{Stream} \mathrel{\tilde{=}} \texttt{gfp}\,(\mathit{Int} \times -)$

– $\mathrm{corec}^{\mathrm{P}} : (A \to \mathit{Int} \times A) \to A \to \mathit{Stream}$

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– $s \oplus t = \mathsf{corec}^{\mathsf{P}}\ (\lambda(s,t).\ ((\mathsf{shd}\ s + \mathsf{shd}\ t)\ ,\ (\mathsf{stl}\ s\ \ ,\ \ \mathsf{stl}\ t)))\ (s,t)$

Primitive Corecursor

`codatatype` $C = \cdots$

– $C \mathrel{\tilde{=}} \texttt{gfp}\ F$

– $\text{corec}^{\text{P}} : (A \to F\ A) \to A \to C$

`primcorec` $f\ \overline{x} = \quad \cdots$

– $f\ \overline{x} = \text{corec}^{\text{P}}\ (\lambda(\overline{x}).\ \cdots)\ (\overline{x})$

(Assuming F is a bounded natural functor)

Smart Corecursor

$$\mathrm{corec}^{P} : (A \to F\ A) \to A \to C$$

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$$\mathtt{corec}\ s \otimes t = (\mathsf{shd}\ s * \mathsf{shd}\ t) \#\# (\mathsf{stl}\ s \otimes t\ \oplus\ s \otimes \mathsf{stl}\ t)$$

– $s \otimes t = \mathsf{corec}_1^{S}\ (\lambda(s,t).$
$\eta\left((\mathsf{shd}\ s * \mathsf{shd}\ t)\ ,\ \eta(\mathsf{stl}\ s,\ t)\ \overline{\oplus}\ \eta(s,\ \mathsf{stl}\ t)\right))\ (s,t)$

– $\overline{\oplus} : \blacksquare\ A \to \blacksquare\ A \to \blacksquare\ A$

– $\eta : A \to \blacksquare\ A$

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– $\overline{\oplus} : \oplus\ A \to \oplus\ A \to \oplus\ A$

– $\eta : A \to \oplus\ A$

$\otimes : C \to C \to C$ has to be friendly

A friendly function can consume
one constructor to produce
at least one constructor.

$\otimes : C \to C \to C$ has to be friendly

$\exists$ parametric $\rho_{\otimes} : (A \times F\ A) \to (A \times F\ A) \to F\ (\otimes\ A)$ s.t.

$$s \otimes t = \cdots\ (\rho_{\otimes}\ (\cdots\ (s, t)))$$

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$$\rho_{\otimes} : (A \times (Int \times A)) \to (A \times (Int \times A)) \to (Int \times \circledast\ A)$$

$$\rho_{\otimes}\ (s, hs, ts)\ (t, ht, tt) = (hs * ht, \eta\ ts\ \overline{\otimes}\ \eta\ t\ \overline{\oplus}\ \eta\ s\ \overline{\otimes}\ \eta\ tt)$$

Isabelle Demo

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Programming Language Examples

Streams and Infinite Trees

```
codatatype 'a stream =
  SCons (shd: 'a)
        (stl: 'a stream)
```

```
up = 0 ## smap (λx. x + 1) up
```

[0,1,2,3,4,5,6,...]

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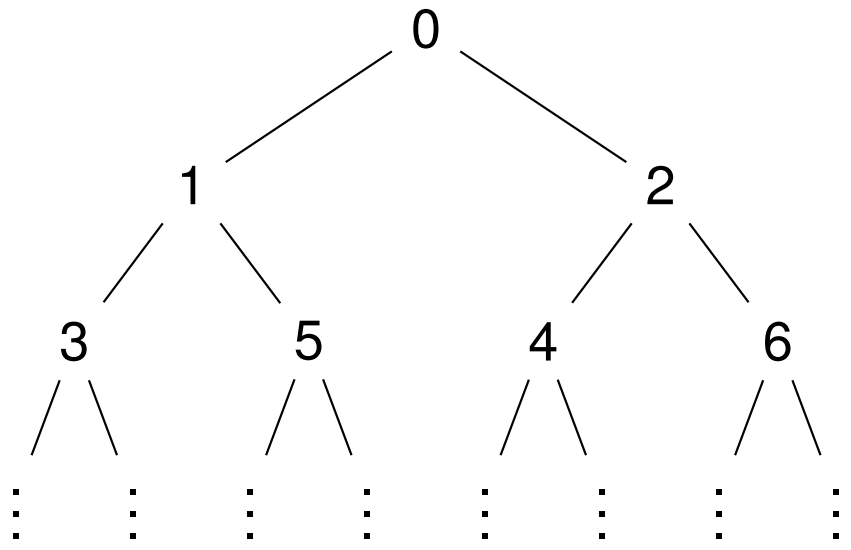

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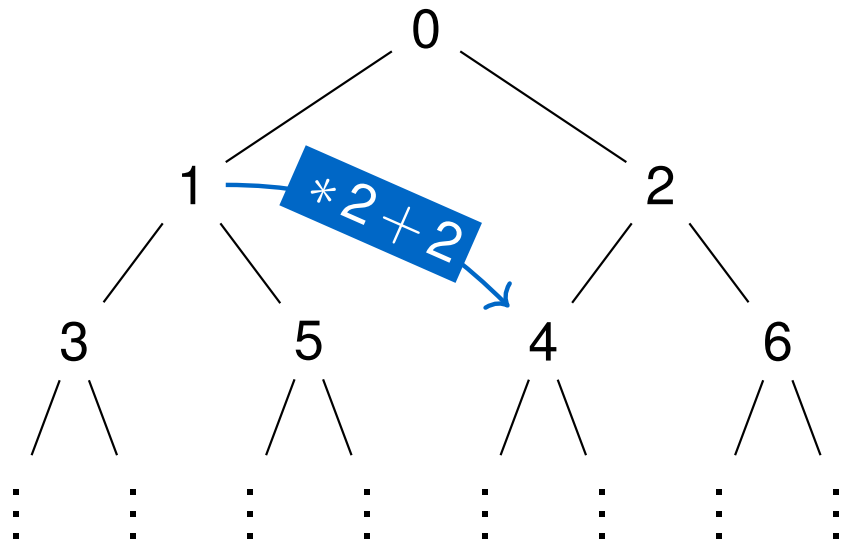

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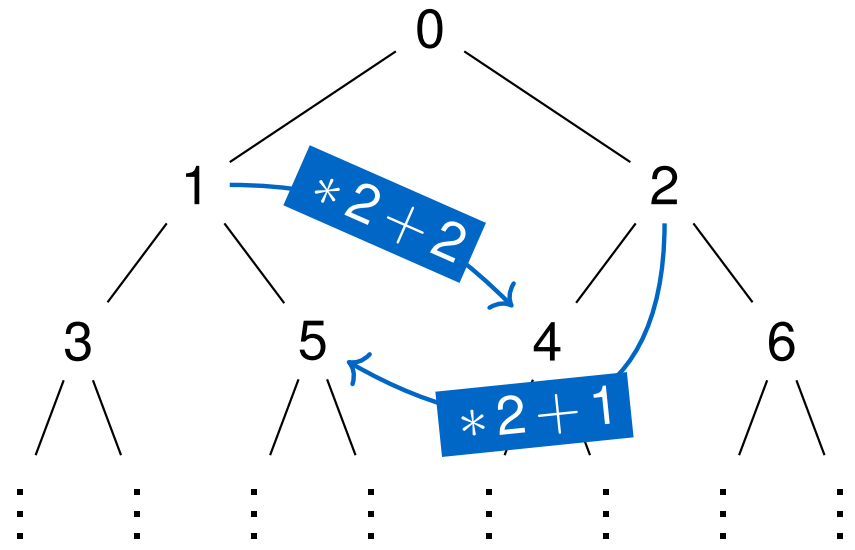



```
tnum = Node (tmap (λx. 2 * x + 1) tnum)
            0 (tmap (λx. 2 * x + 2) tnum)
```

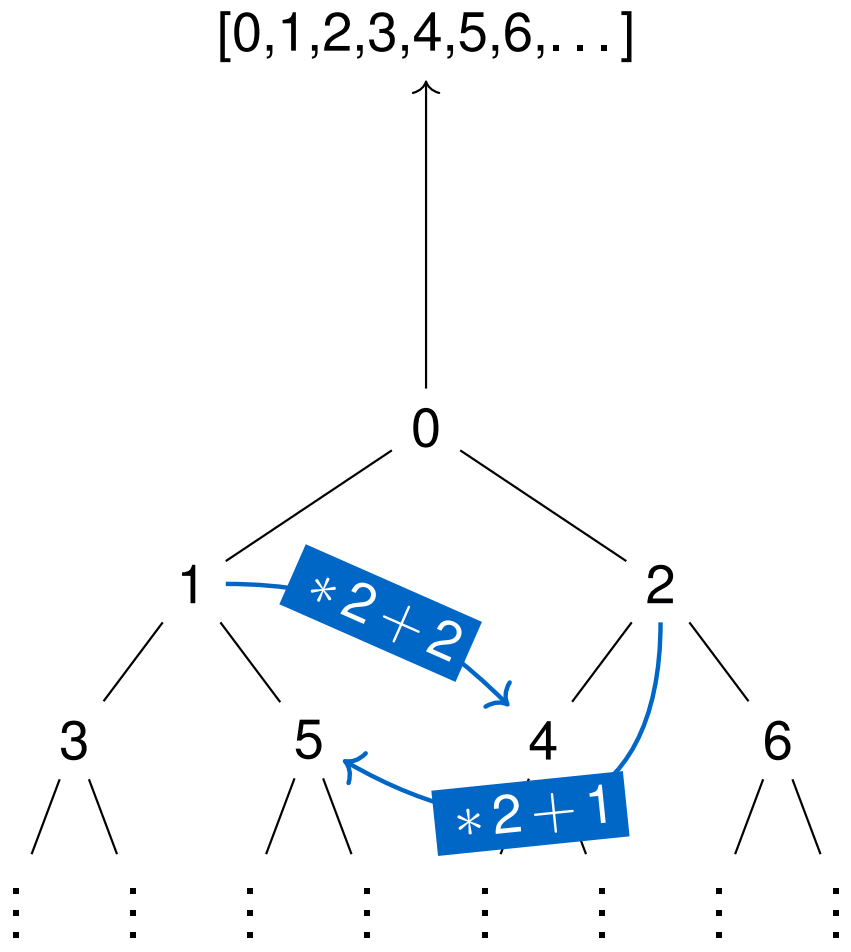
Streams and Infinite Trees

```
codatatype 'a stream =
  SCons (shd: 'a)
        (stl: 'a stream)
```

stream_of

```
codatatype 'a tree =
  Node (left: 'a tree)
       (root: 'a)
       (right: 'a tree)
```

```
up = 0 ## smap (λx. x + 1) up
```

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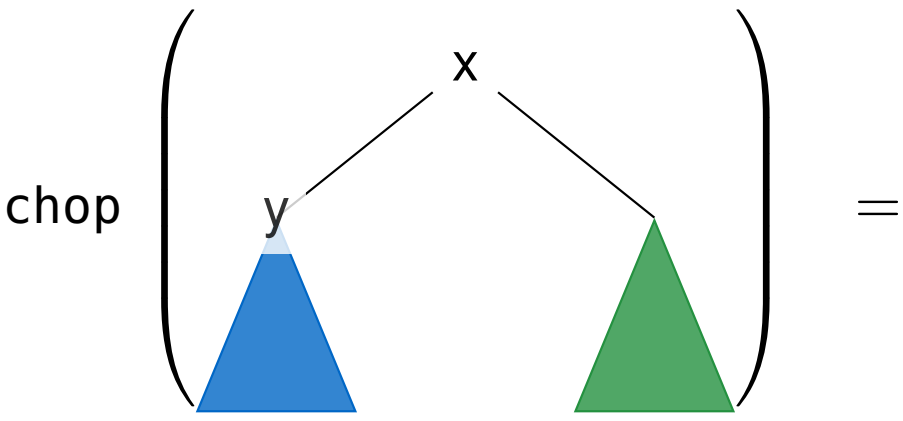
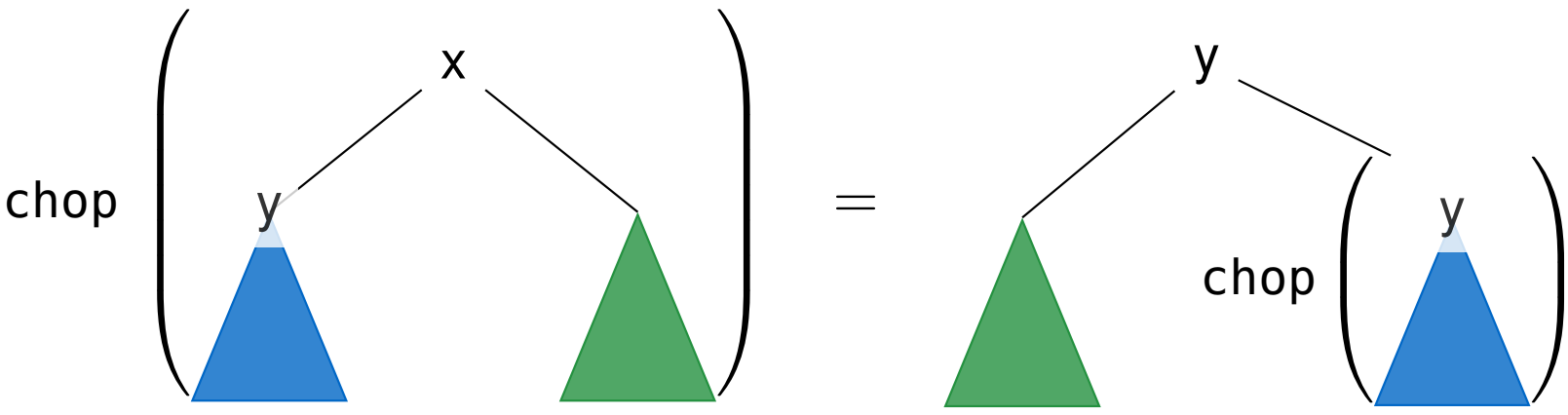
Tree chopping

Remove the root of a tree:



Tree chopping

Remove the root of a tree:



Coinductive traces for a WHILE langauge

Syntax

com ::= SKIP
| ATOM atom
| com ;; com
| IF test com com
| WHILE test com com

Semantic domain

step: state update or test result

trace: [step* | state] or step$^\infty$

```
codatatype trace
  = Final state
  | Step step trace
```

process: state $\Rightarrow$ trace

Coinductive traces for a WHILE langauge

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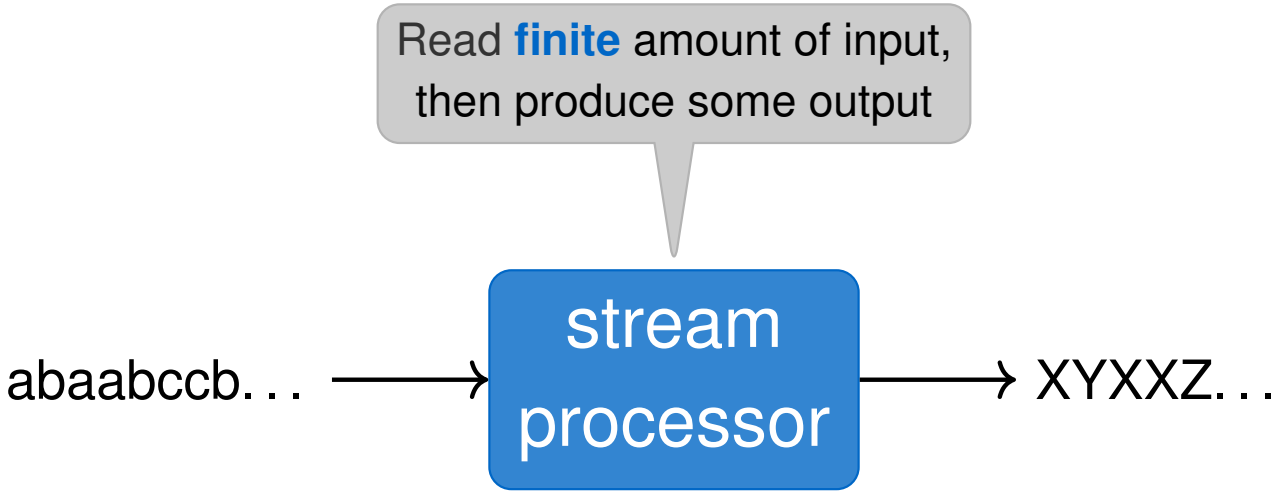
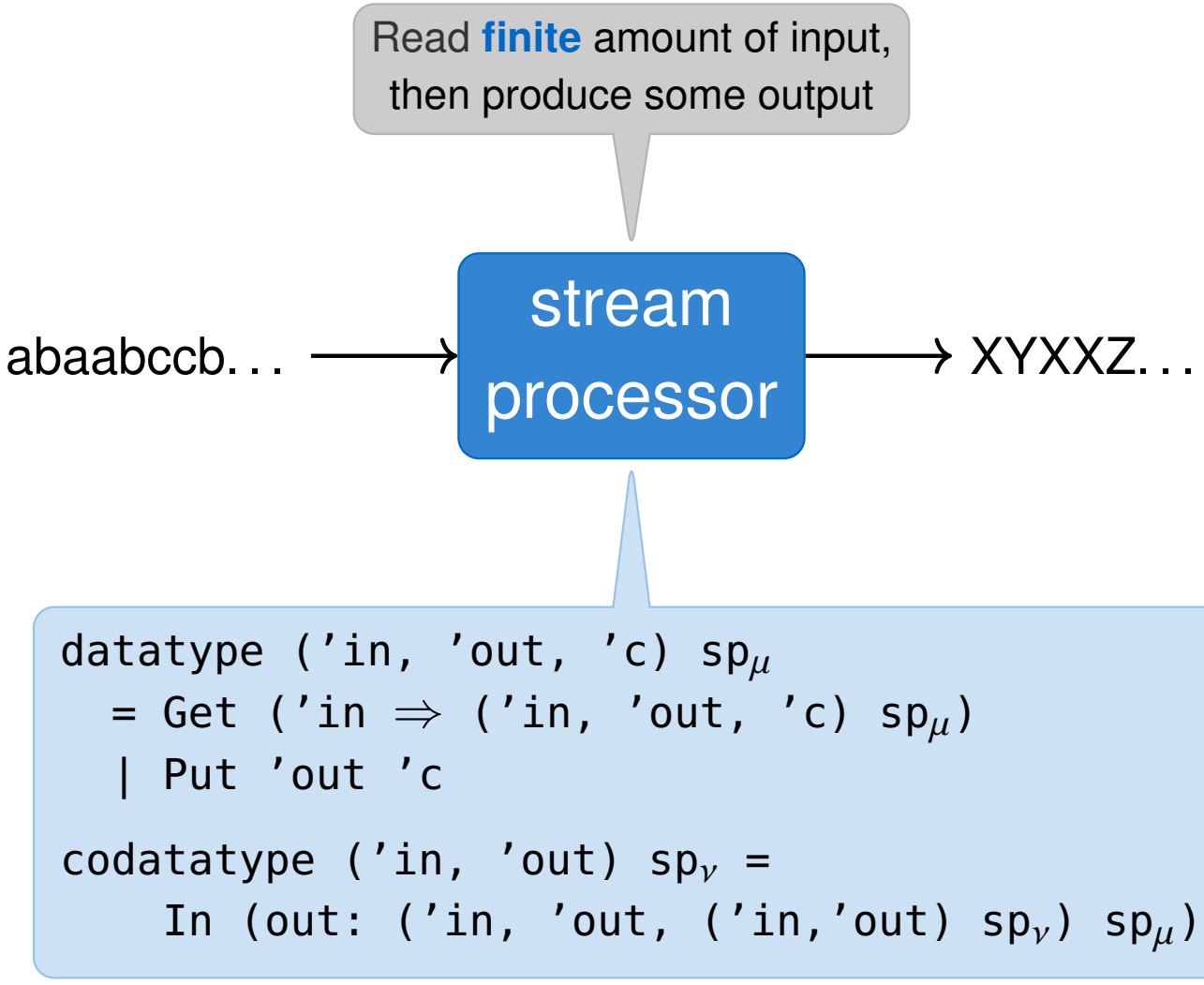
Trace semantics

$[\![\_]\!]$ :: com $\Rightarrow$ process
$[\![$ SKIP $]\!]$ = (λs. [| s])
$[\![$ ATOM atom $]\!]$ = (λs. [Update s | evalA atom s])
$\vdots$

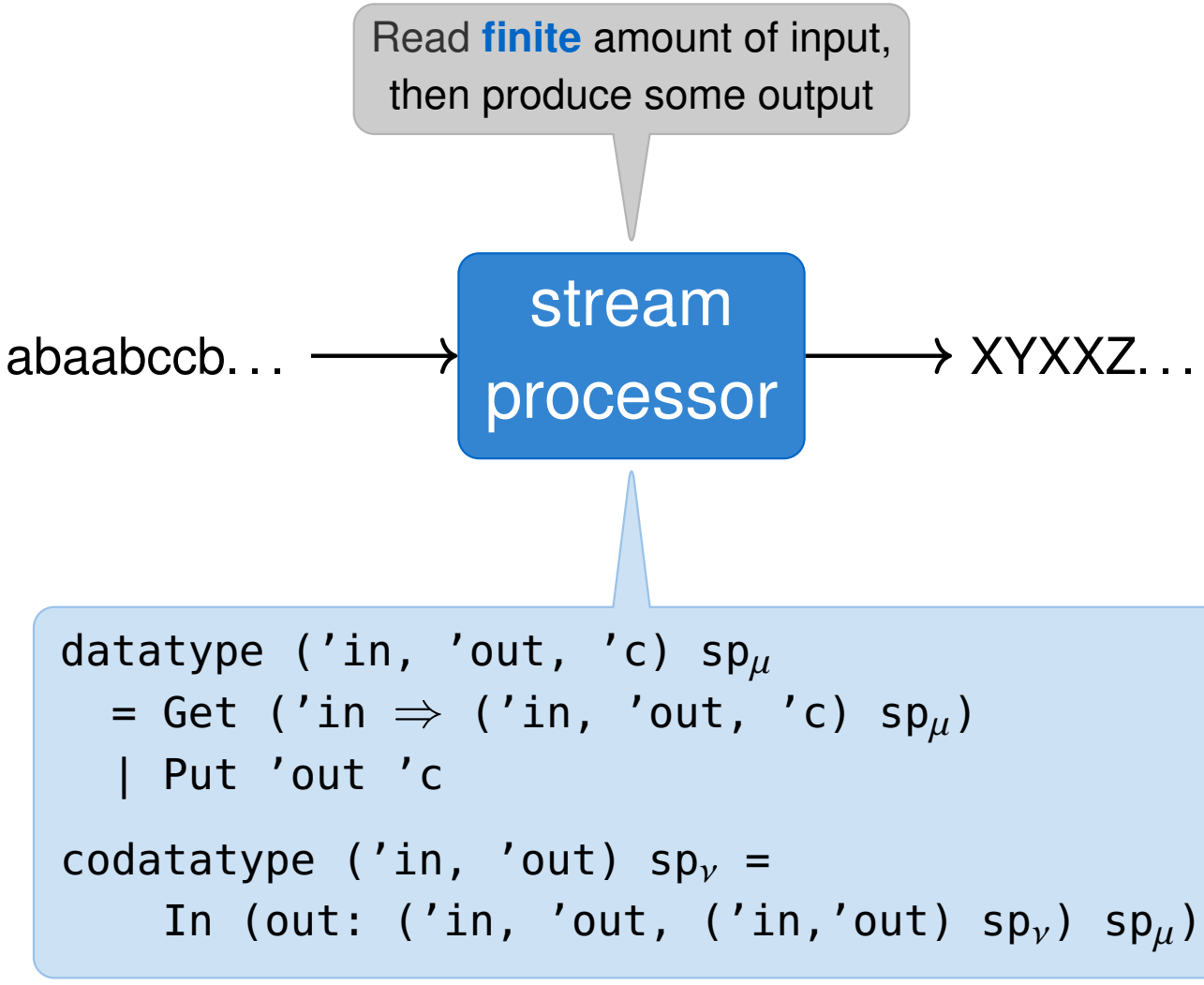
Stream processors



Stream processors



Stream processors

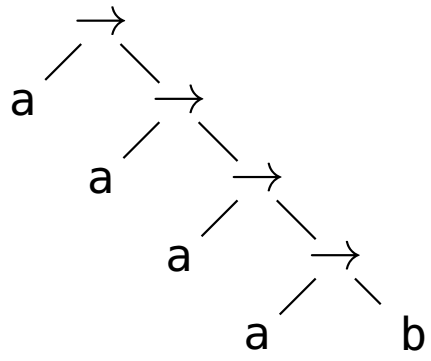


Calculator example by Hur et al. [POPL 2013]

Equirecursive types

Semantics: Infinite types

```
codatatype type
  = TVar var
  | type → type
```



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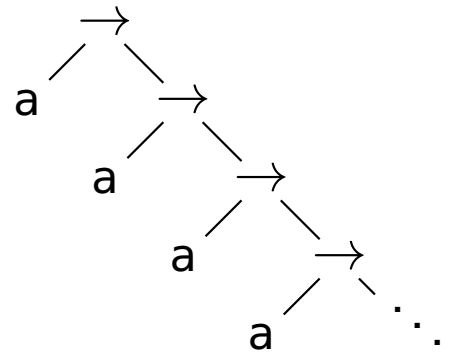

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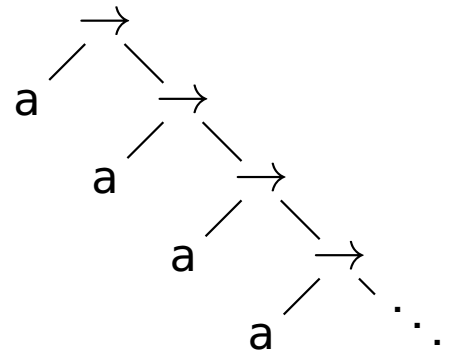


Syntax: Recursive types

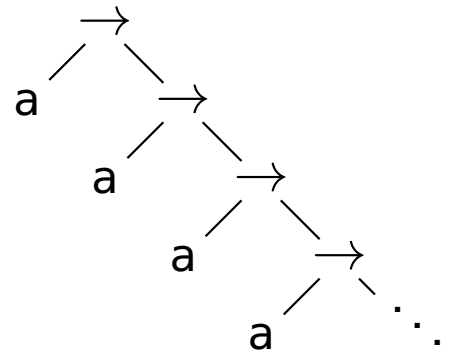
```
datatype ty
  = Var var
  | Arrow ty ty
  | Mu var ty
```

```
Mu X. Arrow (Var a) (Var X)
```

Equirecursive types

Semantics: Infinite types

```
codatatype type
  = TVar var
  | type → type
```



$$[\![\_]\!] :: \text{ty} \Rightarrow \text{type}$$

$$[\![\text{Mu X T}]\!] = \left[\text{X} \mapsto [\![\text{Mu X T}]\!]\right] \cdot [\![\text{T}]\!]$$

Syntax: Recursive types

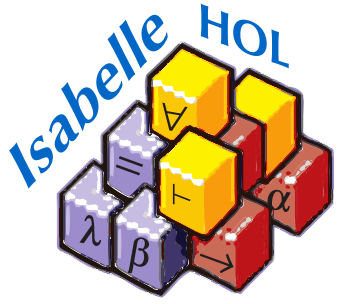
```
datatype ty
  = Var var
  | Arrow ty ty
  | Mu var ty
```

```
Mu X. Arrow (Var a) (Var X)
```

Outlook

- **Stern-Brocot tree of rational numbers**
  `http://www.isa-afp.org/entries/Stern_Brocot.shtml`
- **Knuth-Morris-Pratt string matching**
  [ESOP 2017]
- **Regular languages via Brzozowski derivatives**
  [FSCD 2016]
- **Probabilistic reactive programming**
  CryptHOL
  TLS in Isabelle

Programming and Reasoning with Infinite Data in



Thank You